COMPUTER SOLUTION OF A KINETIC EQUATION FOR ELECTRONS

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Physical and mathematical approaches are presented for the behavior of a weakly ionized plasma in a thermoelectronic converter. Numerical solutions are obtained by computer methods. The distribution function for the electrons is examined in series form for a Boltzmann kinetic equation subject to boundary conditions; the coefficients of the series are deduced via moment equations. The electric field is incorporated in the quasineutrality approximation. An example envisaging only electron-atom collisions is presented.

Consider two unbounded planar electrodes (cathode and anode) heated to different temperatures, between which lies a weakly ionized plasma subject to a potential difference. From the electrodes flow ion and electron fluxes into the plasma, where ionization and recombination can occur. The quantities to be determined are the current, the potential distribution, the temperature, and the charge density. This problem occurs for a cesium converter in the arc mode. If the volume ionization can be neglected, the processes are closely described by the diffusion theory [1], but it is desirable to have more detailed information about the distribution function for the electrons when ionization, excitation, and recombination become important. The diffusion theory is then replaced by a Boltzmann kinetic equation, but this greatly increases the computational difficulties. The present approach envisages the use of computers.

The method of solution is basically as follows. The electron-distribution function in the kinetic equation is replaced by a series in some complete set of functions of the velocity coordinates. There is a second system of independent functions; these are multiplied by the two parts of the kinetic equation, whereupon integration over velocity space gives differential equations of first order in the spatial coordinates for the parameters of the series for the distribution function. These are balance equations or equations for the moments with respect to the above system of independent functions (usually these are polynomials in the velocity coordinates).

We select from this system a subsystem of functions, which we multiply by the boundary conditions for the kinetic equation and integrate over the region where they are given (i.e., with respect to the velocity of the electrons leaving the electrode). This gives the boundary conditions for the differential equations for the moments.

Varieties of this method are to be seen in Grad's [2] and Weitzsch's [3] methods in gas dynamics, or the method of spherical harmonics [4,5] in neutron physics; see [6] for review. The method of expansion used here differs from Grad's method in that I use functions of the energy and spherical angles in velocity space, whereas Grad used functions of the cartesian coordinates of the velocity. Moreover, the zero-th-approximation function is taken as the isotropic $\exp(-mv^2/2kT)$ instead of Grad's anisotropic $\exp[-m(v - v_0)/2kT]$ (m is electron mass, T is temperature, k is Boltzmann's constant, v is particle velocity, and v_0 is the mean particle velocity). These differences are introduced for the following reasons. The electrons in a weakly ionized plasma collide frequently with neutral atoms, so there is more rapid relaxation in momentum than in energy [7], and the distribution function differs little from isotropic. On the other hand, a principal purpose here is to examine the inelastic processes of ionization and excitation, and the major feature is the energy distribution of the electrons without reference to the orientation of the momentum vector. Hence we need take only the first two terms in the expansion with respect to the spherical coordinate $\mu = v_x/v$ (the P_1 approximation in the method of spherical harmonics).

We also take account of the electric field set up by the space charge.

Let d be the distance between cathode and anode, V be the potential differences, r the Debye radius, n_+ and n_- the concentrations of ions and electrons, and q the charge on an electron. As in [1], we consider the case where the main change in the electrical potential U occurs near the electrodes in regions of scale r, while the rest of the region obeys the quasineutrality condition $|n_+ - n_-| \ll n_+$ (Fig. 1).

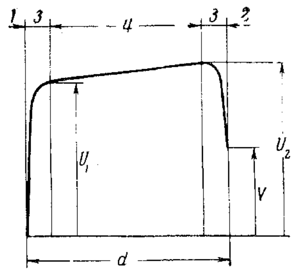

Fig. 1

For this we must have

$$V \leqslant 4\pi n_+ q d^2, \quad \text{or} \quad \frac{qV}{kT} \leqslant \left(\frac{d}{r}\right)^2. \qquad (0.1)$$

The size of the space-charge regions is less than the mean free path for any of the bulk processes, so no scattering occurs in these regions, while their presence is allowed for by the additional potential barriers U_1 (cathode) and $U_2 - V$ (anode). Both physical conditions are obeyed for r sufficiently small.

We also assume that the potential changes monotonically in the space-charge regions, as in Fig. 1, where 1 is the cathode, 2 is the anode, 3 are the space-charge regions, and 4 is the quasineutral plasma.

1. Equations for moments and boundary conditions.

The Boltzmann integrodifferential equation is

$$\frac{\partial f(\mathbf{v}, x)}{\partial t} + v_x \frac{\partial f(\mathbf{v}, x)}{\partial x} + \frac{qF(x)}{m}\frac{\partial f(\mathbf{v}, x)}{\partial v_x} = K(f). \qquad (1.1)$$

Here $f(\mathbf{v}, x)$ is the distribution function for the electrons, v is the velocity of an electron at point x at time t, $F = dU/dx$ is the electric field for an electron, and $K(f)$ is the collisional term, which takes account of the various elastic and inelastic processes, being expressed via integrals with respect to the velocity variables. The boundary conditions are as follows:

cathode $(x = 0)$

a) $v_x f(0, v_x, v_y, v_z) = v_x f_k(\sqrt{v_x^2 - 2qU_1/m}, v_y, v_z)$

for $v_x \geqslant \sqrt{2qU_1/m}$, if $U_1 \geqslant 0$ $\qquad v_x \geqslant 0$, if $U_1 \leqslant 0$,

b) $v_x f(0, v_x, v_y, v_z) = v_x f(0, -v_x, v_y, v_z)$

for $\sqrt{2qU_1/m} > v_x \geqslant 0$, if $U_1 \geqslant 0$; $\qquad (1.2)$

anode $(x = d)$

a) $v_x f(d, v_x, v_y, v_z) = v_x f_a(\sqrt{v_x^2 - 2q(U_2 - V)/m}, v_y, v_z)$

for $v_x \leqslant -\sqrt{2q(U_2 - V)/m}$, if $U_2 - V \geqslant 0$ $\qquad (1.3)$

$v_x \leqslant 0$, if $U_2 - V \leqslant 0$,

$$\text{b) } v_x f(d, v_x, v_y, v_z) = v_x f(d, -v_x, v_y, v_z)$$
$$\text{for } -\sqrt{2q(U_2 - V)/m} < v_x \leqslant 0, \text{ if } U_2 - V \geqslant 0. \quad (1.3) \text{ (cont'd)}$$

Here $f_k(v_x, v_y, v_z)$, $f_a(v_x, v_y, v_z)$ are the distribution functions for the electrons emitted by cathode and anode, which are assumed to be maxwellian:

$$f_k(v_x, v_y, v_z) = N_{k-}\left(\frac{m}{2\pi kT_k}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT_k}\right),$$

$$f_a(v_x, v_y, v_z) = N_{a-}\left(\frac{m}{2\pi kT_a}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT_a}\right). \quad (1.4)$$

Here T_k and T_a are the temperatures of cathode and anode. We convert (1.1)–(1.3) to the variables

$$\varepsilon = \frac{mv^2}{2}, \quad \mu = \frac{v_x}{v}, \quad \varphi = \operatorname{arc\,tg} \frac{v_y}{v_z}. \quad (1.5)$$

The φ dependence drops out because of the axial symmetry in velocity space. In place of (1.1) we have

$$\frac{\partial f(\varepsilon, \mu)}{\partial t} + \frac{\sqrt{2}}{\sqrt{m}}\left[\mu \sqrt{\varepsilon} \frac{\partial f(\varepsilon, \mu)}{\partial x} + \right.$$
$$\left. + qF\left(\mu \sqrt{\varepsilon} \frac{\partial f(\varepsilon, \mu)}{\partial \varepsilon} + \frac{(1-\mu^2)}{2\sqrt{\varepsilon}} \frac{\partial f(\varepsilon, \mu)}{\partial \mu}\right)\right] = K(f). \quad (1.6)$$

We expand the distribution function as a series in μ:

$$f(\varepsilon, \mu) = f_0(\varepsilon) + \mu f_1(\varepsilon) + \ldots . \quad (1.7)$$

The equations for the coefficients in (1.7) are found by substituting (1.7) into (1.6), multiplying by μ^i ($i = 0, 1, \ldots, m$), where m is the number of terms in (1.7), and integrating with respect to μ from −1 to +1.

The boundary conditions are found by multiplying the boundary conditions of (1.2) and (1.3) by μ^i ($i = 0$), $2, \ldots, m-2$, with m even) followed by integration with respect to μ, from 0 to 1 at the cathode and from −1 to 0 at the anode. These equations are equivalent to those in the method of spherical harmonics [4,5] if we ignore the field in the volume and the barriers at the boundaries.

Now we take only two terms in (1.7) (the P_1 approximation of the method of spherical harmonics), in which case the equations for $f_0(\varepsilon)$ and $f_1(\varepsilon)$ become as follows (the time derivative is omitted, as we are interested in the steady state):

$$\frac{2}{3}\frac{\sqrt{2}}{\sqrt{m}}\left[\sqrt{\varepsilon}\frac{\partial f_1(\varepsilon)}{\partial x} + qF\left(\sqrt{\varepsilon}\frac{\partial f_1(\varepsilon)}{\partial \varepsilon} + \frac{f_1(\varepsilon)}{\sqrt{\varepsilon}}\right)\right] = \int_{-1}^{1} K(f)\, d\mu,$$

$$\frac{2}{3}\frac{\sqrt{2}}{\sqrt{m}}\left[\sqrt{\varepsilon}\frac{\partial f_0(\varepsilon)}{\partial x} + qF\sqrt{\varepsilon}\frac{\partial f_0(\varepsilon)}{\partial \varepsilon}\right] = \int_{-1}^{1} K(f)\,\mu d\mu. \quad (1.8)$$

The boundary conditions are as follows:

cathode $\quad (^1/_2 f_0(\varepsilon) + {}^1/_3 f_1(\varepsilon)) =$

$$= \begin{cases} [^1/_2 f_0(\varepsilon)\,\varepsilon^{-1} qU_1 - {}^1/_3 f_1(\varepsilon)\,(\varepsilon^{-1} qU_1)^{3/2} + \\ + {}^1/_2 f_k(\varepsilon - qU_1)\,(1 - \varepsilon^{-1} qU_1) \text{ for } \varepsilon \geqslant qU_1 \geqslant 0, \\ [^1/_2 f_0(\varepsilon) - {}^1/_3 f_1(\varepsilon)] \quad \text{for } 0 \leqslant \varepsilon < qU_1, \\ ^1/_2 f_k(\varepsilon - qU_1) \quad \text{for } qU_1 \leqslant 0; \end{cases} \quad (1.9)$$

anode $\quad (^1/_2 f_0(\varepsilon) - {}^1/_3 f_1(\varepsilon)) =$

$$= \begin{cases} [^1/_2 f_0(\varepsilon)\,\varepsilon^{-1} q(U_2 - V) + {}^1/_3 f_1(\varepsilon)\,(\varepsilon^{-1} q(U_2 - V))^{3/2} + \\ + {}^1/_2 f_a(\varepsilon - q(U_2 - V)) \times [1 - \varepsilon^{-1} q(U_2 - V)] \\ \text{for } \varepsilon \geqslant q(U_2 - V),\ q(U_2 - V) \geqslant 0, \\ [^1/_2 f_0(\varepsilon) + {}^1/_3 f_1(\varepsilon)] \quad \text{for } 0 \leqslant \varepsilon < q(U_2 - V), \\ ^1/_2 f_a(\varepsilon - q(U_2 - V)) \quad \text{for } q(U_2 - V) \leqslant 0. \end{cases} \quad (1.10)$$

Here we have used the fact that f_k and f_a are independent of μ. We expand $f_0(\varepsilon)$ and $f_1(\varepsilon)$ as functions of the energy:

$$f_0(\varepsilon) = \left(\frac{m}{2\pi kT_0}\right)^{3/2} \exp\left(-\frac{\varepsilon}{kT_0}\right) \sum_{i=0}^{n} A_i(x)\, L_i\left(\frac{\varepsilon}{kT_0}\right), \quad (1.11)$$

$$f_1(\varepsilon) = \left(\frac{m}{2\pi kT_0}\right)^{3/2} \exp\left(-\frac{\varepsilon}{kT_0}\right) \sum_{i=0}^{n} B_i(x)\, L_i\left(\frac{\varepsilon}{kT_0}\right). \quad (1.12)$$

The calculation based on retaining n + 1 terms in (1.11) and (1.12) is termed the n-th approximation. The functions $L_i(\varepsilon/kT_0)$ are polynomials of order i, while T_0 (the expansion parameter) may depend on x.

We substitute (1.11) and (1.12) into (1.8)–(1.10) and multiply (1.8) by the function

$$\frac{3\pi^{3/2}}{m\,(kT_0)^{1/2}}\,\varepsilon\left(\frac{2}{m}\right)^{1/2}\left(\frac{\varepsilon}{kT_0}\right)$$

and (1.9) and (1.10) by

$$\frac{4\sqrt{2}\pi^{3/2}}{m^{3/2}(kT_0)^{1/2}}\,\varepsilon L_i\left(\frac{\varepsilon}{kT_0}\right)$$

and integrate with respect to $d\varepsilon$ from 0 to ∞. This gives us 2(n + 1) first-order differential equations for $A_i(x)$ and $B_i(x)$ and n + 1 boundary conditions at the cathode and at the anode. The equations are

$$\sum_{j=0}^{n} K_{ij}\frac{dB_j(x)}{dx} +$$
$$+\left(-\frac{qF}{kT_0}N_{ij} + \frac{d\ln T_0}{dx}S_{ij}\right)B_j(x) = J_a^{\,i}(A, B), \quad (1.13)$$

$$\sum_{j=0}^{n} K_{ij}\frac{dA_j(x)}{dx} + \left(-\frac{qF}{kT_0}M_{ij} + \frac{d\ln T_0}{dx}S_{ij}\right)A_j(x) =$$
$$= J_b^{\,i}(A, B) \quad (i = 0, 1, \ldots, n), \quad (1.14)$$

$$K_{ij} = \int_0^\infty e^{-x} L_i(x)\, L_j(x)\, x dx,$$

$$M_{ij} = \int_0^\infty e^{-x}\,(L_j(x) - L_j'(x))\, L_i(x)\, x dx,$$

$$N_{ij} = \int_0^\infty e^{-x}\,[(L_j(x) - L_j'(x))\, x - L_j(x)]\, L_i(x)\, dx,$$

$$S_{ij} = \int_0^\infty e^{-x}\left[x^2 (L_j(x) - L_j'(x)) - \frac{3}{2} L_j(x)\, x\right] L_i(x)\, dx,$$

$$J_a^{\,i}(A, B) = \frac{3\pi^{3/2}}{m\,(kT_0)^{1/2}} \int_0^\infty d\varepsilon \int_{-1}^{1} d\mu K(f)\, L_i\left(\frac{\varepsilon}{kT_0}\right)\sqrt{\varepsilon},$$

$$J_b^{\,i}(A, B) = \frac{3\pi^{3/2}}{m\,(kT_0)^{1/2}} \int_0^\infty d\varepsilon \int_{-1}^{1} d\mu K(f)\, L_i\left(\frac{\varepsilon}{kT_0}\right)\mu\sqrt{\varepsilon}. \quad (1.15)$$

The boundary conditions for (1.13) are as follows:

cathode

$$\gamma_i(f_k, U_1, T_0(0)) = \sum_{j=0}^{n}\left[\alpha_{ij}\left(\frac{qU_1}{kT_0(0)}\right)A_j + \frac{2}{3}\beta_{ij}\left(\frac{qU_1}{kT_0(0)}\right)B_j\right]$$

anode

$$\gamma_i(f_a, U_2 - V, T_0(d)) = \sum_{j=0}^{n}\alpha_{ij}\left[\left(\frac{q(U_2-V)}{kT_0(d)}\right)A_j - \frac{2}{3}\beta_{ij}\left(\frac{(U_2-V)q}{kT_0(d)}\right)B_j\right]. \quad (1.16)$$

Here

$$\gamma_i(f, U, T) = \begin{cases} \left(\frac{2\pi kT}{m}\right)^{3/2}\int\limits_{qU}^{\infty} f(\varepsilon - qU)\times \\ \times\left(\frac{\varepsilon}{kT} - \frac{qU}{kT}\right)\frac{d\varepsilon}{kT} \quad (U \geqslant 0), \\ \left(\frac{2\pi kT}{m}\right)^{3/2}\int\limits_{0}^{\infty} f(\varepsilon - qU)\frac{\varepsilon}{kT}\frac{d\varepsilon}{kT} \quad (U \leqslant 0), \end{cases}$$

$$\alpha_{ij}(y) = \begin{cases} \int\limits_{y}^{\infty}\exp(-x)L_i(x)L_j(x)(x-y)\,dx \quad (y \geqslant 0), \\ \int\limits_{0}^{\infty}\exp(-x)L_i(x)L_j(x)x\,dx \quad (y \leqslant 0), \end{cases}$$

$$\beta_{ij}(y) = \begin{cases} 2\alpha_{ij}(0) + \int\limits_{y}^{\infty}\exp(-x)L_i(x)L_j(x)\times \\ \times\left(\left(\frac{y}{x}\right)^{3/2} - 1\right)x\,dx \quad (y \geqslant 0), \\ \alpha_{ij}(0) \quad (y \leqslant 0). \end{cases}$$

Consider the $K(f)$ term for the case where there are only collisions of electrons with neutral atoms. Taking the atoms as infinitely heavy, we have

$$K(f) = n_a v\int(f(\varepsilon, \mu') - f(\varepsilon, \mu))\,\sigma(\theta, \varepsilon)\sin\vartheta\, d\vartheta\, d\varphi$$
$$\left(v = \frac{\sqrt{2\varepsilon}}{\sqrt{m}}\right). \quad (1.17)$$

Here n_a is the density of the atoms, v is the velocity of an electron, μ and μ' are the coordinates of that velocity before and after collision (ε being unchanged), ϑ and φ are the angles for the rotation of the velocity vector as a result of collision, and $\sigma(\vartheta, \varepsilon)$ is the differential cross-section.

Substitution of (1.7), (1.11), and (1.12) in (1.17), and of (1.17) into (1.15), gives the following expressions for the right halves of (1.13):

$$J_a^i(A, B) = 0, \quad J_b^i(A, B) = \sum_{j=0}^{n} K_{ij}B_j\frac{1}{L}$$
$$\left(\frac{1}{L} = n_a Q,\ Q = 2\pi\int\limits_{0}^{\pi}(1 - \cos\vartheta)\,\sigma(\vartheta)\sin\vartheta\, d\vartheta\right). \quad (1.18)$$

Here Q is the cross-section for momentum transfer, which is assumed independent of the energy.

Equation (1.14) gives K_{ij}. Allowance for other types of collision causes (1.18) to contain the corresponding functions of the coefficients of the series (electron-electron collisions and recombination give a form quadratic in A_i and B_i, while ionization and inelastic electron-atom collisions give a linear form).

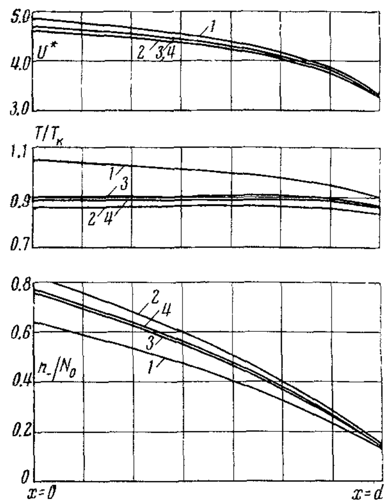

Fig. 2

The integrodifferential kinetic equation in partial derivatives is thus reduced to a system of ordinary first-order differential equations.

If this system is solved in the approximation n = 1, while $T_0(x)$ is put equal to the electron temperature at the given point, system (1.13) becomes equivalent to the equations of diffusion and thermal conduction for the electrons in [1], the first two boundary conditions of (1.16) having the following* physical meaning: the number and energy of the electrons escaping from an electrode and passing through a plane parallel to it and directly adjacent to the barrier are equal to the total number and energy of the emitted electrons plus the number reflected by the barrier.

2. Convergence in energy space and choice of polynomial system. Consider the series of (1.11) for the isotropic part of the distribution function in the n-th approximation.

The A_i are deduced by solving the equations of the previous section; they are dependent on the choice of the $L_i(x)$, but substitution into (1.11) for the A_i determined for different $L_i(x)$ gives equivalent expressions dependent only** on n.

We use Laguerre polynomials of index 1/2 as our L_i; these are orthogonal with the weight $\exp(-x)x^{1/2}$:

$$\int\limits_{0}^{\infty} e^{-x}x^{1/2}L_i(x)L_j(x)\,dx = \delta_{ij}i!\,\Gamma(i + 3/2). \quad (2.1)$$

*These boundary conditions differ from those of [1] in that the latter involved neglect of the contribution from the anisotropic part of the distribution function in calculating the current and the energy of the electrons escaping from the plasma. This contribution is in fact small when the diffusion theory is used.

**The choice of the $L_i(x)$ is not indifferent as regards the best organization of the numerical calculations.

Then n_- and kT for the f_0 of (1.11) are defined by the first two coefficients:

$$n_- = A_0, \quad T = T_0 (1 + A_1 / A_0). \qquad (2.2)$$

The A_i of (1.11) are dependent on the order n of the approximation. Let $f_0^*(\varepsilon)$ be the exact solution of (1.8). We may say that for $n \to \infty$ the A tend to the limit

$$A_i^* = \left(\frac{2\pi}{m}\right)^{3/2} \left(\Gamma\left(i + \frac{3}{2}\right) i!\right)^{-1} \int_0^\infty f_0^*(\varepsilon) L_i\left(\frac{\varepsilon}{kT_0}\right) \sqrt{\varepsilon}\, d\varepsilon. \qquad (2.3)$$

Here the A_i^* are the coefficients in the expansion with respect to the $L_i(\varepsilon/kT_0)$ in a space with weight $\rho(\varepsilon)$ for $\varphi(\varepsilon)$:

$$\rho(\varepsilon) = \sqrt{\varepsilon} \left(\frac{m}{2\pi kT_0}\right)^{3/2} \exp\frac{-\varepsilon}{kT_0}, \qquad \varphi(\varepsilon) = \sqrt{\varepsilon}\, \frac{f_0^*(\varepsilon)}{\rho(\varepsilon)}.$$

To get the $f_0^*(\varepsilon)$ for $n \to \infty$ we must have that the series for $\varphi(\varepsilon)$ converges. This requires finiteness in

$$R = \qquad (2.4)$$

$$= \int_0^\infty \rho(\varepsilon) |\varphi(\varepsilon)|^2 d\varepsilon = \left(\frac{2\pi kT_0}{m}\right)^{3/2} \int_0^\infty \exp\frac{\varepsilon}{kT_0} (f_0^*(\varepsilon))^2 \varepsilon^{1/2} d\varepsilon.$$

Let the asymptotic behavior of $f_0^*(\varepsilon)$ be as $\exp(-\varepsilon/kT_*)$; then R is finite for $T_* < 2T_0$. The effects of the field and of collisions that produce a redistribution of the electron energy (electron–electron and inelastic electron–atom collisions) become small for $\varepsilon \to \infty$, so the energy distribution becomes as for the emission from the hotter electrode (always the cathode), so we may put $T_* = T_k$.

Thus the convergence is dependent on T_0. The diffusion theory is equivalent to n = 1, and there T_0 is chosen so that kT_0 is the mean energy of an electron. Extending this method of choosing T_0, we get from (2.2) the condition*

$$A_1 = 0. \qquad (2.5)$$

It is quite possible that this choice of T_0 causes the condition $T_k < 2T_0$ to be violated at some point; then the series is divergent** for $n \to \infty$, although a reasonable result may be obtained for n small (the diffusion theory for n = 1).

*This may be given the following variational interpretation: T_0 must give the extreme value of

$$\int_0^\infty \ln \rho(T_0, \varepsilon) f_0 \sqrt{\varepsilon}\, d\varepsilon,$$

which resembles the expression for the entropy

$$\int_0^\infty (\ln f) f \sqrt{\varepsilon}\, d\varepsilon.$$

**But it can be made convergent by truncating the distribution function for the electrons escaping from the cathode at some finite energy. This cutoff energy may be arbitrarily large, so, in principle, in that case we may obtain an exact solution.

We can choose T_0 in a different way be requiring that

$$A_n = 0. \qquad (2.6)$$

Conditions (2.5) and (2.6) coincide for n = 1, so they are both possible generalizations of the way of choosing T_0 in the diffusion theory. The asymptotic behavior of $f_0^*(\varepsilon)$ is as $\exp(-\varepsilon/kT_k)$. It can be shown that the coefficients in the series are dependent solely on the asymptotic behavior for n sufficiently large, so the coefficients of $f_0^*(\varepsilon)$ will be proportional to those of $\exp(-\varepsilon(kT_k)$, and these are proportional to

$$\left(1 - \frac{T}{T_0}\right) \frac{1}{n!}.$$

Hence condition (2.6) for $n \to \infty$ is equivalent to $T_0 = T_k$, so the A_n tend to 0 and the series converges.

Consider now the case in which we allow only for the collision of electrons with infintely heavy atoms, while U_1 and $U_2 - V$ are positive (reflect electrons from the plasma); some of the electrons are trapped in a potential well. This problem has been considered in papers on these converters [8, 9]; the initial integrodifferential equation then has many solutions. In fact, in this case we cannot neglect the electron energy changes that always occur in collisions, or the lateral loss of electrons; but the solution to the equations for the moments of section 1 is unique for any n.

The A_i and B_i for $n \to \infty$ correspond to a single distribution function for the electrons; to derive this function unambiguously from the initial integrodifferential equation we must find the collisional term, which leads to energy relaxation for the electrons. We allow the corresponding relaxation time to tend to infinity to get in the limit a solution to the integrodifferential equation that gives the desired distribution.

3. Allowance for space-charge field in the quasi-neutrality approximation. Equations (1.13) and (1.16) with condition (2.5) define the electron distribution via $A_i(x)$ and $B_i(x)$ for a known potential distribution. To determine the latter we need to know the charge-density distribution. Here the diffusion approximation is sufficient, taking the ion temperature equal to the atom temperature [1], the latter being determined from the equation of thermal conduction for a gas (the effects of the electrons and ions on the neutral gas may be neglected, as the plasma is only weakly ionized)

$$T_{an}^{1/2} \frac{dT_{an}}{dx} = 0, \qquad (3.1)$$

in which $T_{an}(x)$ is the temperature of the atoms. The boundary conditions give the solution as

$$T_{an}(x) = T_k \left(1 + \left[\left(\frac{T_a}{T_k}\right)^{3/2} - 1\right] \frac{x}{d}\right)^{2/3}, \qquad (3.2)$$

in which T_k and T_a are the temperatures of cathode and anode.

The ion current I_+ is [10] given by

$$I_+ = -qD_+ \left(\frac{dn_+}{dx} + (k_t + 1) n_+ \frac{1}{T_{an}} \frac{dT_{an}}{dx} + \frac{qF}{kT_{an}} n_+\right). \qquad (3.3)$$

The conservation of charge gives

$$\frac{dI_+}{dx} = -\frac{dI_-}{dx}. \qquad (3.4)$$

The electron current I_- is expressed via the B_i, while the diffusion coefficient D_+ may be put as $D_+ = L_+ v_+/3$, in which L_+ is the mean free path for an ion, whose thermal velocity is v_+. We assume that $k_t = 0$ for thermal diffusion (Chapman and Cowling state that this corresponds to diffusion of particles in a gas when the particles and atoms are of the same size). The boundary conditions for (3.3) and (3.4) become as for electron diffusion in the approximation n = 0 (with allowance for the change in the sign of the barrier for the ions and for the relation of A_0 to n_+ and of B_0 to I_+).

We express n_+ in (3.3) via (2.2) and the quasineutrality condition

$$n_+ (x) = n_- (x). \qquad (3.5)$$

Then (3.3) may be used to find F = dU/dx, while the boundary values U_1 and U_2 for U are determined by the boundary conditions for (3.3) and (3.4).

This method of allowing for the field of the space charge will be rigorous if the conditions stated in the introduction are obeyed. To test whether the solution corresponds to these conditions, we must estimate $n_- - n_+$ for the region of the quasineutral plasma from Poisson's equation

$$d^2U/dx^2 = 4\pi q (n_- - n_+).$$

It is also necessary to solve Poisson's equation for the regions near the electrodes. The boundary values are the potentials at the boundary of the plasma (i.e., U_1 and U_2) together with the known electrode potentials, while the size of the region is chosen to be such that dU/dx becomes zero at the boundary with the plasma (this is equivalent to the condition for continuity in dU/dx for $r \to 0$).

4. Numerical solution. The problem is that of solving (1.13), (3.3), and (3.4) subject to the boundary conditions. From (2.5) and (3.5), the unknown functions are

$$U_0 (x),\ T_0 (x),\ A_0 (x),\ A_2 (x), \ldots,$$
$$A_n (x),\ I_+ (x),\ B_0 (x), \ldots,\ B_n (x).$$

In general we may put that

$$\frac{dy_i (x)}{dx} = f_i (x, \ldots, y_j \ldots) \quad (i, j = 1, \ldots, 2m). \qquad (4.1)$$

The boundary conditions are

$$F_k (\ldots y_i (0) \ldots) = 0 \quad (k = 1, \ldots, m),$$
$$F_{k+m} (\ldots y_i (d) \ldots) = 0. \qquad (4.2)$$

These and the differential equations are nonlinear, the former being given at both ends of an interval, x = 0 and x = d.

Equations (4.2) are considered as 2m nonlinear equations in the functions $y_i(0)$ at x = 0; the $y_i(d)$ are deduced from the $y_i(0)$ by numerical integration of (3.6) by the Runge-Kutta method, Newton's method [11] being used to solve the nonlinear system.

The following example takes account only of electron-atom collisions for the case mentioned at the end of section 2: at cathode and anode there are barriers that reflect electrons. Electron or ion emission from the anode is neglected. Figure 2 shows the results for n of 1, 2, 3, and 4 for the following values of the dimensionless potential:

$$\frac{qV}{kT_k} = 0$$
$$\frac{T_a}{T_k} = 0.544, \quad \ln\left(\frac{N_{k+}}{N_{k-}}\right)^{1/2} = 4.6$$
$$\frac{d}{L_-} = \frac{d}{L_+} = 7.2.$$

Here N_{k+} and N_{k-} are the densities of the emission fluxes of ions and electrons at the cathode [see (1.4)], while L_+ and L_- are the mean free paths of those particles at the cathode. The ordinates are the dimensionless quantities

$$T / T_k,\ u^* = qU / kT,\ n_-/N_0,$$
$$(N_0 = \sqrt{N_{k+} N_{k-}}).$$

Here N_0 is the equilibrium charge concentration at the cathode. Ten steps were used in the Runge-Kutta integration from x = 0 to x = d; doubling the step for n = 2 altered only the fourth significant figure.

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